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MULTIPLE BAND DIFFERENCE OPERATOR-MATRICES WITH RETURNED SEQUENCES

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Abstract. In this paper we study the behavior of the sequence $\{a_n\}$ of complex numbers satisfying the relation $a_{n+k} = q_1a_n + q_2a_{n+1} + ... + q_ka_{n+k-1}$, where $\{q_m\}$ is a fixed sequence of complex numbers. Such kind of sequences arise in problems of analysis, fixed point theory, dynamical systems, theory chaos and etc. For example, investigating the spectra of triple and more than triple band triangle operator-matrices arise above mentioned sequences which required to study the behavior of the sequence. Till now the received formulas for the spectra of considered operator-matrices from the point of application looks like very complicated. In this work for the eliminating of indicated flaws we apply new approach where the formulas for the spectra describe circular domains.

Keywords: spectrum, difference operator-matrices, generalized difference operator-matrices, sequence spaces, returned sequences, circular domains.

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1 Introduction

Here we summarize the important knowledge in the existing literature concerning the spectra difference operator - matrices and their generalizations. The spectrum and fine spectrum of the difference operator-matrix Δ over the sequences spaces c_0 and c has been studied by Altay and Basar (2005). Akhmedov and Basar (2006; 2007) have studied the fine spectrum of the difference operator-matrix Δ over the sequences spaces l_p and bv_p, where $1 = p < 8$. Note that the sequence space $bv_p(1 = p < 8)$ was studied by Basar and Altay (2003) and Akhmedov and Basar (2007). Malafosse (2002) has studied the spectrum and fine spectrum of the operator Δ over the sequences spaces s_r where s_r denotes the Banach space of all numerical sequences $x = (x_n)$ normed by

$$\|x\|_{s_r} = \sup_{n\in N} \frac{|x_n|}{r^n} \quad (r > 0)\,.$$

The fine spectrum of the Zweirer operator-matrix Z^s over the spaces l_1 and bv has been investigated by Altay and Karakus (2005). Now we give results concerning to the spectra of some generalizations of difference operator-matrices. The fine spectrum of the generalized double-band operator $B(r, s)$ over the sequence spaces c_0 and c has been studied by Altay and Bashar (2005). Also, the fine spectrum of the operator $B(r, s)$ over the sequence spaces l_1 and bv has been examined by Furkan et al. (2006). The fine spectrum of the operator $B(r, s)$ over the sequence spaces l_p and bv_p, where $(1 < p < \infty)$ has been determined by Bilgic and Furkan, (2008). The fine spectrum of the generalized difference operator Δ_ν over the sequences spaces c_0 and l_1 was investigated by Srivastava and Kumar (2009; 2010). In Akhmedov & El-Shabrawy, (2010; 2011) the authors have proved by the counter examples that some of their results are incorrect and the corresponding corrected ones have been provided. The fine spectrum of the operator Δ_ν over the sequences spaces c has been examined in Akhmedov and El-Shabrawy, (2010; 2011). The definition of the operator Δ_ν have been modified and the fine spectrum have been determined for the modified operator Δ_ν over the sequence spaces c and l_p, where $(1 < p < \infty)$. The fine spectrum of the generalized difference operator $\Delta_{a,b}$ over the sequences spaces c_0, c and l_1 has been studied in Akhmedov and El-Shabrawy, (2010; 2011; 2015). The spectrum of the upper triangular double-band matrices over the sequence spaces c_0 and c has been determined by Karakaya and Altun (2010). The fine spectrum of the triple-band matrices $B(r,s,t)$ over c_0 and c has been examined by Furkan et al (2007), over the sequence spaces l_1 and bv by Bilgiç and Furkan, (2007) and over the sequence spaces l_p and bv_p $(1 < p < \infty)$ by Furkan et.al (2010). The spectrum of the triangular operator-matrix $D(r,o,s,o,t)$ has been examined by Tripathy and Paul, (2013) the spectrum and fine spectrum of generalized second order forward difference operator $\Delta^2_{u,v,w}$ on the sequence spaces l_1 have been studied by Panigrahi & Srivastava, (2012) and etc.

Note that the formulas of spectrum for double-band matrices usually describe circular domains.

But for triple and more triple-band matrices the receiving formulas of spectra from the point of application looks like very complicated. We will return to this issue.

We recall some basic concepts of spectral theory which are needed for our investigation.

Let X be a Banach space and $T : X \to X$ be a bounded linear operator. By $R(T)$ we denote the range of T, i.e.

$$R(T) = \{y \in X : y = Tx, x \in X\}.$$

By $B(X)$ we denote the set off all bounded linear operators on X into itself. If $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the topological dual X^* of X defined by $(T^*f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Let $X \neq \{\theta\}$ be a complex normed space and if $D(T)$ is a domain of definition of the linear operator $T : D(T) \to X, D(T) \subseteq X$, with T we associate the operator $T_\lambda = T - \lambda I$, where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse which is linear, we denote it by. $T_\lambda^{-1} = (T - \lambda I)^{-1}$ and call it the resolvent operator T. Many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties.

Definition 1. *Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. A regular value λ of T is a complex number such that*

(R1) T_λ^{-1} exists,

(R2) T_λ^{-1} is bounded,

(R3) T_λ^{-1} is defined on a set which is dense in X.

The resolvent set of T, denoted by $\rho(T)$, is meant $\sigma(T,X) = C/\rho(T,X))$ in the complex plane C is called the spectrum of T. Furthermore, the spectrum of $\sigma(T,X)$ is partitioned into three disjoint sets as follows:

The point (discrete) spectrum $\sigma(T,X)$ is the set such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T,X)$ is called an eigenvalue of T.

The continuous spectrum $\sigma_c(T,X)$ is the set such that T_λ^{-1} exists and satisfies (R3), but not (R2) that is, T_λ^{-1} is unbounded.

The residual spectrum $\sigma_r(T,X)$ is the set such that T_λ^{-1} (may be bounded or not) but does not satisfy (R3), that is, the domain of T_λ^{-1} is not dense in X.

2 On the returned sequences

In this section we study some iterative processes arising in analysis spectral theory of linear operators, in fixed point theory, theory of dynamical systems, theory chaos and etc. (Bilgic et al., 2007; Bilgic & Furkan, 2007; Furkan, 2010; Tripathy & Paul, 2013; Panigrahi & Srivastava, 2012; Kreyszing, 1978; Slavisa, 1965; Khan et al, 2012). We shall analyze special class of iterative processes which we call returned sequences of real or complex numbers. By returned sequence of order k we mean the sequence which it's every term is subordinated with k number preceding terms. In other words

$$a_{n+k} = q_1 a_n + q_2 a_{n+1} + ... + q_k a_{n+k-1}, \tag{1}$$

where (a_n) is a sequence of complex numbers, $q_1, q_2, ..., q_k$ giving complex numbers. For example,

$$a_{n+2} = q_1 a_n + q_2 a_{n+1}. \tag{2}$$

The sequence of the form (2) is a second order returned sequence. The arithmetic progression

$$a_{n+1} = a_n + d$$

is a second order returned sequence. Really, from relations $a_{n+2} = a_{n+1} + d$, $a_{n+1} = a_n + d$ we get that $a_{n+2} = -a_n + 2a_{n+1}$. Another example, $\left(n^2\right)$ is a third order returned sequence. Really, $a_{n+3} = a_n - 3a_{n+1} + 3a_{n+2}$, where $a_n = n^2$.

From the point of view the returned sequence of the order k is a solution of the linear homogeneous difference equation of the order k with constant coefficients. Setting first k terms of returned sequence is equivalent to setting initial conditions of the Cauchy problem of the difference equation.

Theory of linear homogeneous equations with constant coefficients sometimes allows to find the general term of the returned sequence. But in many cases to do it very difficult. If in (2) $q_1 = 0$ and $|q_2| < 1$,then the sequence (a_n) is a geometric progression. Note that from the point of applications the next lemma present interest.

Lemma 1. *(Akhmedov & El-Shabrawy, 2010; Tripathy & Das, 2015). Let (c_n) and (d_n) be two sequences of complex numbers such that $\lim\limits_{n\to\infty} c_n = c$ and $|c| < 1$. Define the sequence (z_n) of complex numbers such that $z_{n+1} = c_{n+1} z_n + d_n$ for all $n \in N_0 = N \cup (0)$. Then the next assertions are true:*

(i) (z_n) is bounded, iff (d_n) is bounded;

(ii) (z_n) is convergent, iff (d_n) is convergent;

(iii) (z_n) is a null sequence, iff (d_n) is a null sequence.

It is clear that if $d_n \equiv 0$, then the sequence is a second order returned sequence. In general if the order of the returned sequence greater than two studying the behavior of such sequences requires great effort.

Let us explain it for the case (2). Dividing both side of (2) by a_{n-1} $(a_n \neq 0)$ we get

$$\frac{a_{n+1}}{a_{n-1}} = q_1 \frac{a_n}{a_{n-1}} + q_2. \tag{3}$$

Denote

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = L.$$

Then from (3) we have

$$L^2 - q_1 L - q_2 = 0.$$

Last equation is called a characteristic equation of (2). There were attempts to study the behavior of the sequence (a_n) for the case (2), where every term (beginning from third step) is subordinated with two preceding terms. If number of subordinated numbers are three, the order of the characteristic equation will be three, and if four, then order will be four, and so on. But for these cases we can use this method until not more forth order equation excluding some special cases. The fact is that beginning from fifth order in general cases we don't have any general form for the roots in radical of the characteristic equation. Even for third order equation to analyze the roots is very hard.

We offer new method for the investigation the behavior of the sequence (a_n) with condition (1) not using characteristic equation. The following theorem is important and will be used in the proof of theorems given in the present section.

Theorem 1. *Let (a_n) be a sequence of complex numbers, k is a natural number and $q_1, q_2, ..., q_k$ are given complex numbers such that $|q_1| + |q_2| + ... + |q_k| < 1$ and $a_{n+k} = q_1 a_n + q_2 a_{n+1} + ... + q_{k-1} a_{n+k-2} + q_k a_{n+k-1}$, then $(a_n) \in l_p$ $(1 \le p \le \infty)$.*

Proof. Let us denote

$$S'_n = \sum_{m=1}^{n} |a_m| .$$

Using the relation (1) we get the next system

$$a_{1+k} = q_1 a_1 + q_2 a_2 + ... + q_k a_k$$

$$a_{2+k} = q_1 a_2 + q_2 a_3 + ... + q_k a_{k+1} \tag{4}$$

$$\dots\dots\dots\dots\dots\dots$$

$$a_{n+k} = q_1 a_n + q_2 a_{n+1} + ... + q_k a_{n+k-1}.$$

Taking module of both sides of this system and adding them we have

$$|a_{1+k}| + |a_{2+k}| + ... + |a_{n+k}| \le |q_1| \cdot (|a_1| + ... + |a_n|) + |q_2| \cdot (|a_2| + ... + |a_{n+1}|) + ...+ \\ + |q_k| \cdot (|a_k| + |a_{k+1}| + ... + |a_{n+k-1}|) .$$

Adding to the both sides the missing terms and increasing right sides necessary terms we get.

$$|a_1| + |a_2| + ... + |a_k| + |a_{1+k}| + |a_{2+k}| + ... + |a_{n+k}| \le \\ \le |a_1| + |a_2| + ... + |a_k| + |q_1| \cdot (|a_1| + ... + |a_n| + |a_{n+1}| + |a_{n+2}| + ... + |a_{n+k}|) + \\ + |q_2| \cdot (|a_1| + |a_2| + ... + |a_{n+1}| + |a_{n+2}| + ... + |a_{n+k}|) + ...+ \\ + |q_k| \cdot (|a_1| + ... + |a_{k-1}| + |a_k| + |a_{k+1}| + ... + |a_{n+k-1}| + |a_{n+k}|) .$$

or

$$S'_{n+k} \le (|a_1| + |a_2| + ... + |a_k|) + (|q_1| + |q_2| + ... + |q_k|) \, S'_{n+k}.$$

From it we may have

$$S'_{n+k} \le \frac{|a_1| + |a_2| + ... + |a_k|}{1 - (|q_1| + |q_2| + ... + |q_k|)}. \tag{5}$$

We see that the monotonic increasing positive sequences (S'_n) bounded and therefore it converges absolutely.

Suppose

$$S' = \lim_{n \to \infty} S'_n = \sum_{n=1}^{\infty} |a_n| .$$

From (5) it follows that

$$|a_n| \le A \cdot q^n, \tag{6}$$

where $A = |a_1| + |a_2| + ... + |a_n|$, $q = |q_1| + |q_2| + ... + |q_n|$. Hence, $(a_n) \in l_1$.

Now using (6) and the known inequality

$$\left(\sum_{m=1}^{k} |b_m| \right)^p \le k^{p-1} \sum_{m=1}^{k} |b_m|^p \quad (1 \le p < \infty)$$

for any numbers b_m $(m = 1, 2, \ldots, k)$ we get

$$|a_{n+k}|^p \leq k^{p-1} \left(|q_1|^p \cdot |a_{n-1+k}|^p + |q_2|^p \cdot |a_{n-2+k}|^p + ... + |q_k|^p \cdot |a_n|^p\right).$$

Now repeating the proof of the present theorem for l_1 we can show that $(a_n) \in l_p$ $(1 \leq p \leq \infty)$. Theorem is completely proved. □

Corollary 1. *Under the condition of theorem 1* $a_n = B \cdot q_0^n$, $B = const$,

$$q_0 = q_1 + q_2 + ... + q_k.$$

Proof. Denote

$$S_n = \sum_{m=1}^{n} a_m, \quad S = \sum_{m=1}^{\infty} a_m.$$

From the relation (4) we have

$$\begin{gathered} S_{n+k} - (a_1 + a_2 + ... + a_k) = q_1 S_n + q_2 S_{n+1} + ... + q_k S_{n+k-1} - q_2 a_1 - q_3 (a_1 + a_2) - \\ -q_4 (a_1 + a_2 + a_3) - ... - q_k (a_1 + a_2 + ... + a_{k-1}) \end{gathered} \tag{7}$$

By the theorem the sequences $\{S_n\}$ converges, $\lim\limits_{n\to\infty} S_n = S$. From (7) we have

$$(1 - (q_1 + q_2 + ... + q_k))\, S = a_1 + a_2 + ... + a_k - q_2 a_1 - q_3 (a_1 + a_2) - ... - q_k (a_1 + a_2 + ... + a_{k-1})$$

or

$$S = \frac{B}{1 - q_0}, q_0 = q_1 + q_2 + \ldots q_k,$$

$$B = a_1 + a_2 + ... + a_k - q_2 a_1 - q_3 (a_1 + a_2) - ... - q_k (a_1 + a_2 + ... + a_{k-1}).$$

We know that $|q_0| \leq |q_1| + |q_2| + ... + |q_k| < 1$ (by the condition of theorem 1). Then $S = B \cdot \sum q_0^n$ and $a_n = B \cdot q_0^n$. This completes the proof of corollary. □

3 Recent results

In this section our aim to review some recent results concerning the spectrum of the more than that double band (triple, quadruple, and etc.) generalized difference operator-matrices acting in some sequence spaces. In such works (Bilgiç et al., 2007; Bilgiç & Furkan, 2007) have been used the method where the main role plays the analyzing of the roots of characteristic equations of returned sequences of the order k $(k \geq 2)$.

Denote by c_0, c, l_∞ and bv (or bv_1) the null, convergent, bounded and bounded variation sequences spaces, respectively. Also by l_p $(1 \leq p \leq \infty)$ and bv_p $(1 \leq p < \infty)$, we denote the spaces of all p-absolutely summable and p-bounded variation sequences spaces, respectively. Main focus in the works (Bilgiç et al., 2007, Bilgiç & Furkan, 2007) was the triple-band matrix $B(r, s, t)$, where

$$B(r, s, t) = \begin{bmatrix} r & 0 & 0 & 0 & ... \\ s & r & 0 & 0 & ... \\ t & s & r & 0 & ... \\ 0 & t & s & r & ... \\ ... & ... & ... & ... & ... \end{bmatrix},$$

s and t are complex parameters which do not simultaneously vanish.

The next results were received.

Theorem 2. $B(r, s, t) \in B(c_0) \quad (B(c))$, *and*

$$\|B(r, s, t)\|_{(c_0, c_0)} = \|B(r, s, t)\|_{(c, c)} = |r| + |s| + |t|.$$

Theorem 3. *$B(r,s,t) \in B(l_p) \quad (B(bv_p))$ and*

1. $(|t|^p + |\delta|^p + |t|^p)^{1/p} \le \|B(r,s,t)\|_{l_p} \le |r| + |s| + |t|.$

2. $\|B(r,s,t)\|_{bv_p} \le |r| + |s| + |t|.$

Choosing the square roots of s the next theorem have been proved.

Theorem 4. *(Bilgiç et al., 2007, Theorem 2.1, 2.10 and 2.11) Let s be a complex number such that $\sqrt{s^2} = -s$ and define the set S by*

$$S = \left\{ \lambda \in C : \left| \frac{2(r-\lambda)}{-s + \sqrt{s^2 - 4t(r-\lambda)}} \right| \le 1 \right\}.$$

Then $\sigma(B(r,s,t), X) = S$, where X is one of the sequences c_0, c, l_p $(1 \le p \le \infty)$ and bv_p $(1 \le p < \infty)$, respectively.

The same results were received in (Tripathy & Paul, 2013, lemma 3, lemma 4, theorem 5) for the artificial generalization of difference operator-matrix in c_0 and c.

Let's now discuss the works (Panigrahi & Srivastava, 2012; Vandajav & Undrakh, 2014), where the above indicated questions were investigated for some triple band generalizations of the difference operator-matrices, where nonconstant diagonals are different numerical sequences. For example, the linear operator $\Delta_{\upsilon uw} : c_0 \to c_0$ is defined by

$$\Delta_{\upsilon uw} x = (\upsilon_n x_n + u_{n-1} x_{n-1} + w_{n-2} x_{n-2})_{n=0}^{\infty}, \quad \text{with} \quad x_{-1} = x_{-2} = 0, \quad \text{where} \quad x = (x_n) \in c_0.$$

It is easy to see that the operator-matrix $\Delta_{\upsilon uw}$ has the next matrix form

$$\Delta_{\upsilon uw} = \begin{bmatrix} \upsilon_0 & 0 & 0 & 0 & 0 & 0 & \dots \\ u_0 & \upsilon_1 & 0 & 0 & 0 & 0 & \dots \\ w_0 & u_1 & \upsilon_2 & 0 & 0 & 0 & \dots \\ 0 & w_1 & u_2 & \upsilon_3 & 0 & 0 & \dots \\ 0 & 0 & w_2 & u_3 & \upsilon_4 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

Unfortunately, in both works (Panigrahi & Srivastava, 2012; Vandajav & Undrakh, 2014) as continuation of the work Srivastava & Kumar, (2009) made many mistakes and as a consequence received incorrect results. Also in both indicated works were used the method with characteristic equation for the triple band matrix $\Delta_{\upsilon uw}$. Let's cite the next theorem.

Theorem 5. *Assume $\sqrt{u^2} = u$ and define the set S by*

$$S = \left\{ \lambda \in C : \frac{2|\upsilon - \lambda|}{\left| -u + \sqrt{u^2 - 4w(\upsilon - \lambda)} \right|} \le 1 \right\},$$

where $\lim_{n\to\infty} w_n = w \ne 0$, $\lim_{n\to\infty} \upsilon_n = \upsilon \ne 0$, $\lim_{n\to\infty} u_n = u \ne 0$, $w_n \ge 0$, $u_n \ge 0$ and (υ_n) be a constant or strictly decreasing sequence of nonnegative numbers.

Then the spectrum of the operator $\Delta_{\upsilon uw}$ on c_0 is given by $\sigma(\Delta_{\upsilon uw}, c_0) = S$.

4 New Approach for the investigation the spectrum of the operator $B\left(r,s,t\right)$ in c_0, c, l_p and bv_p $\left(1\leq p\leq\infty\right)$

In this subsection as an example for the operator $B\left(r,s,t\right)$ we show that it's spectrum describe circular domains in complex plane C.

For this we shall use theorem 1 of section 2.

As we indicated above our main focus here is on the matrix-operator $B\left(r,s,t\right)$. We assume here and hereafter that r,s and t are complex parameters, such that they denote simultaneously vanish. It is known that operator $B\left(r,s,t\right)$ is a linear bounded operator (acting in c_0,c,l_p $(1\leq p\leq\infty)$, bv_p $(1\leq p<\infty)$), which were calculated (see theorem 2, theorem 3) of the section 3. The following lemmas will be used in the proofs of theorems given in the present section.

Lemma 2. *Let $1<p<\infty$ and $A\in B(l_1)\bigcap B(l_\infty)$, then $A\in B(l_p)$.*

Lemma 3. *A linear operator T has a dense range if and only if T the adjoint T^* of T is one to one.*

Theorem 6.

$$\sigma(B\left(r,s,t\right),l_p)=\sigma(B(r,s,t),c_0)=\sigma(B(r,s,t),c)=\sigma(B(r,s,t),l_\infty)=\sigma(B(r,s,t),bv_p)D,$$

where

$$D=\{\sigma\in C\colon |\lambda-r|\ \leq|s|+|t|\}.$$

Proof. It is sufficient to prove the theorem for the space l_1. First we prove that $(B\left(r,s,t\right)-\lambda I)^{-1}$ exists and is in $B(l_1)$ for $\lambda\notin D$ and the operator $B\left(r,s,t\right)-\lambda I$ is not invertible for $\lambda\in D$. Formally we can calculate that

$$\left(B\left(r,s,t\right)-\lambda I\right)^{-1}=\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & \dots\\ a_2 & a_1 & 0 & 0 & 0 & \dots\\ a_3 & a_2 & a_1 & 0 & 0 & \dots\\ a_4 & a_3 & a_2 & a_1 & 0 & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix},$$

where

$$a_1=\frac{1}{r-\lambda},a_2=-\frac{s}{(r-\lambda)^2},a_3=\frac{s^2-(r-\lambda)t}{(r-\lambda)^3},...,a_n=-\frac{s}{r-\lambda}a_{n-1}+\frac{t}{r-\lambda}a_{n-2} \tag{8}$$

for all $n\geq3$. Denote

$$q_1=\frac{-s}{r-\lambda},q_2=\frac{-t}{r-\lambda}.$$

Then from (8) we have

$$a_n=q_1a_{n-1}+q_2a_{n-2} \tag{9}$$

for all $n\geq3$. We see that the sequences (a_n) defining by (9) is a second order returned sequences.

Let $\lambda\notin D$. Then we have

$$|\lambda-r|>|s|+|t|.$$

That is

$$\frac{|s|+|t|}{|\lambda-r|}<1. \tag{10}$$

Now instead last inequality we take

$$\frac{|s|+|t|}{|\lambda-r|}=|q_1|+|q_2|<1.$$

Repeating the proof of theorem 1 and it's corollary $\left\|(B\,(r,s,t)-\lambda I)^{-1}\right\|_{l_1}$ is bounded if and only if holds (10). This shows that $(B\,(r,s,t)-\lambda I)^{-1}\in B\,(l_1)\,.$

Similarly we can show that $(B\,(r,s,t)-\lambda I)^{-1}\in B(l_\infty)$ since $\lambda\in D\Rightarrow|\lambda-r|>|s|+|t|$ or $|q_1|+|q_2|<1$. Using the lemma 2 we assert that $(B\,(r,s,t)-\lambda I)^{-1}\in B(l_p)$ $(1<p<\infty)$.

Using the above reasoning about l_p we may show that the last assertion is true also for the spaces $l_p(1<p<\infty)$. It is easy to show that the assertion of this theorem is true also for the spaces c_0,c and bv_p $(1\le p<\infty)$. Theorem is proved completely. □

Theorem 7.

$$\sigma_p(B(r,s,t),X)=\emptyset,$$

where $X$is one of the spaces c_0,c,l_p and $bv_p(1\le p<\infty)$.

Theorem 8.

$$\sigma(B^*(r,s,t),X^*)=\{\lambda\in C,|\lambda-r|<|s|+|t|\},$$

where $B^*(r,s,t)$ is Banach adjoint of $B(r,s,t)$ and X^* is a dual space of the spaces of c_0c and $l_p(1\le p<\infty)$ respectively.

Proof. The assertion of this theorem follows from known fact that

$$\sigma(A,X)=\sigma(A^*,X^*).$$

for any operator $A\in B(X),X$ is a Banach space and X^* is a dual of X. □

Theorem 9.

$$\sigma_r(B(r,s,t),X)=\{\lambda\in C\colon\ |\lambda-r|<|s|+|t|\},$$

where $X$is one of the space c_0,c,l_p and bv_p $(1<p<\infty)$.

Theorem 10.

$$\sigma_c(B(r,s,t),X)=\{\lambda\in C\colon\ |\lambda-r|=|s|+|t|\}.$$

Proof. Since $\sigma(B(r,s,t),X)$ is the disjoint union of the parts σ_p,σ_r, and σ_c. The assertion of the theorem is true. The theorem is proved.

Similarly we can prove the next theorems. □

Theorem 11.

$$\sigma_r(B(r,s,t),X)=\{\lambda\in C\colon\ |\lambda-r|\le|s|+|t|\},$$

where$X$is one of the space l_1 and bv.

$$\sigma_c(B(r,s,t),X)=\emptyset.$$

Theorem 12.

$$\sigma_r(B(r,s,t),X)=\{\lambda\in C\colon|\ \lambda-r|<|s|+|t|\},$$

where X is one of the spaces of c_0, c, l_p $(1\le p\le\infty)$ and bv_p $(1\le p<\infty)$.

Proof. Since $\sigma(B(r,s,t),X)$ is the disjoint union of the parts

$\sigma_p(B(r,s,t),X)$, $\sigma_c(B(r,s,t),X)$ and $\sigma_r(B(r,s,t),X)$ in Banach space X we must have the assertion of the present theorem. □

5 Conclusion

In this work we summarize recent results concerning the spectral analysis of some generalized difference operator-matrices over some sequence spaces. Especially we take into consideration the triple and more than triple operator-matrices. As it is known the investigation the spectra of triple band and more than triple band difference operator-matrices leads us to the next relation of the type

$$a_{n+k} = q_1 a_n + q_2 a_{n+1} + \ldots + q_k a_{n+k-1},$$

where $\{q_m\}$ is a fixed finite system of complex numbers, $\{a_n\}$ is a sequence of the terms of the solvent of considered operator-matrices. It appears that such kind of relation arise in many problems of not only in analysis and also in fixed point theory, dynamical systems, theory chaos and so on. Therefore to study the behavior of these sequences present scientific interest. In some works were made an attempt to study this problem. No diminishing the value of obtained results we state that the using the "characteristic equation" method may work till not more fourth order inclusive. Even in third order case the formulas of the spectra of considered operator-matrices are very complicated. In present work for the eliminating of indicated above flaws we apply new method allowing to study the spectral problems of any generalized repeated band difference-operator-matrices.

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